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MAGNETOHYDRODYNAMIC NOZZLE FLOW WITH THREE TRANSITIONS

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Magnetohydrodynamic Nozzle Flow with Three Transitions

Abstract

The flow of an infinitely-conducting inviscid gas in the presence of a magnetic field everywhere parallel to the velocity is considered. Such a flow starting from rest and expanding to supersonic velocities has three transitions, across each of which the flow differential equations change type: from elliptic to hyperbolic or vice versa. The flow velocities at the three transitions are, respectively, the sound speed c, the Alfven speed a, and the speed $ca/(c^2 + a^2)^{1/2}$. A class of those flows is approximately constructed by solving the flow differential equations, with the flow velocity etc., prescribed on an axis of symmetry. The solution is in the form of a formal power series in a neighborhood of the axis. The resulting streamlines give possible symmetric nozzle contours. Some mathematical investigations are made to determine the acceptable form of the data to be given on the axis. Numerical examples giving nozzle shapes are included and compared with similar nozzles for non-conducting gases. The procedure described is an extension of that of Friedrichs for ordinary gas nozzles.

1. INTRODUCTION

This paper gives a formal power series solution (near the axis of symmetry) for the plane nozzle flow of an infinitely-conducting inviscid gas with a magnetic field everywhere parallel to the flow velocity. The gas expands from near-reservoir conditions to supersonic velocities in the nozzle.

When an ordinary non-conducting gas expands from rest to supersonic velocities in a de Laval nozzle, it is well-known that the gas crosses a sonic transition (near the throat of the nozzle), upstream of which the flow is subsonic and governed by an elliptic differential equation, and downstream of which the flow is supersonic and governed by a hyperbolic differential equation. Moreover, we know that such a flow is stable, whereas the reverse flow (i.e. supersonic flow slowing down to subsonic flow through continuous compression) is unstable, and thus occurs physically only when accompanied by shocks.

In the case of an infinitely-conducting gas with a magnetic field parallel to the velocity everywhere, it is by now well-known (See e.g. Grad[1]) that if such an expanding flow is possible, the gas must cross three transitions. The differential equation governing the flow is elliptic at low velocities, changes to hyperbolic across the first transition, back to elliptic across the second, and again to hyperbolic across the third. For definiteness, we shall assume that the field strength is such that the gas reaches Alfven speed "a" before it reaches sound speed "c" (although the procedure is general so that assuming the converse is just as good). The first transition occurs at a gas speed equal to $ca/(c^2+a^2)^{1/2}$, the second at "a", and the third at "c". Thus far, we have no definite knowledge that such a fully "transonic" flow exists. Intuition is insufficient; for example, Grad[1] has shown that the addition of a small transverse field component will prevent such flows from crossing the sonic transition, a fact which is intuitively non-obvious. The present paper does not prove the existence or inexistence of such fully "transonic" flows, nor does it answer the question whether such flows are stable. It merely gives an approximate description of such flows in nozzles, their existence being assumed. Nevertheless, the well-behaved properties of the solution tend to suggest the probable existence of these flows, rather than their inexistence.

The procedure used is that of Friedrichs[2] for ordinary gas nozzles. The potential and stream functions are used as independent variables, and the space coordinates and hodograph variables as dependent variables. Data are given analytically on a line (say, the x-axis) to give solutions symmetric with respect to it. This is a Cauchy problem, and the solution is sought in the form of a power series, its existence being guaranteed by the Cauchy-Kowalewski Theorem. Any streamline in the physical plane is then a nozzle contour. Since the region of convergence of the series solution is unknown, this solution is only a formal solution for small distances from the axis of symmetry, with the first term equal to the one-dimensional or "hydraulic" solution. By a similar procedure, Frankl[3] went further to investigate the region of convergence, thus yielding an existence proof in addition to an approximate description for the flow.

Our procedure is essentially the same as that of Friedrichs. We assign a velocity distribution on the axis and obtain the nozzle contour as well as all flow variables in the neighborhood of the axis. However, as will be seen, our data cannot be given as arbitrary analytic functions, because the axis becomes characteristic at one of the transitions.

Section 2 gives the basic differential equations of the flow, and the transformation of variables that are used. Section 3 gives the formal power series solution, in terms of given Cauchy data on the axis. Section 4 is devoted to an investigation of the singularities and the necessary restrictions on the Cauchy data. Finally, we give some numerical examples in Section 5, and discuss the results and conclusions.

2. THE BASIC DIFFERENTIAL EQUATIONS

The basic differential equations for the steady flow of an infinitely conducting, inviscid, compressible fluid (i.e., the steady state Lundquist equations) follow:

Continuity $\operatorname{div}(\rho\bar{q}) = 0$ (1)

Momentum $\frac{\nabla p}{\rho} + (\bar{q}\cdot\nabla)\bar{q} + \frac{\bar{B}}{\mu\rho} \times \operatorname{curl}\bar{B} = 0$ (2)

Energy $(\bar{q}\cdot\nabla)S = 0$ (3)

Equation of State $p = C(S)\,\rho^{\gamma}$ (4)

Maxwell's Equations $\operatorname{curl}(\bar{B} \times \bar{q}) = 0$ (5)

$\operatorname{div}\bar{B} = 0$ (6)

Here ρ is the density, $\bar{q}$ the velocity, p the pressure, $\bar{B}$ the magnetic field, S the entropy, and γ the ratio of specific heats. This is a system of 10 equations in 9 unknowns. Had we considered the time-dependent Lundquist equations, (6) would merely play the role of an admissibility condition on the initial data, (as is well known in Maxwell's equations), and (1) to (5) form a well-defined hyperbolic system. In steady flow, (6) is needed as an additional equation.[4] We consider the special class of such flows (see e.g. Grad[1], Kogan[5], Resler-Sears[6]) where the field and velocity are everywhere parallel[7]. As is done in all these references, we let $\bar{B} = \alpha\rho\bar{q}$. Then (5) would be trivially satisfied. (1) and (6) require that α be chosen constant along each streamline. Moreover, along each streamline, (2) integrates into an ordinary Bernoulli equation

$$\int \frac{dp}{\rho} + \frac{q^2}{2} = E \qquad (7)$$

where E, the total enthalpy, is constant along each streamline.

In this paper, we make the further restriction that each of the three quantities S, α, and E be constant not only along every streamline, but also throughout the flow. This would be the case, of course, if at infinity, a uniform flow with a parallel field exists. Otherwise, this is an additional restriction.

Rewriting the momentum equation, we have

$$\nabla E - \overline{q} \times \text{curl}\, \overline{q} + (\alpha/\mu)\overline{q} \times \text{curl}\, \alpha\rho\overline{q} = 0$$

or $$\overline{q} \times \text{curl}\, (1 - \alpha^2\rho/\mu)\overline{q} = 0$$

This implies that $\text{curl}(1-\alpha^2\rho/\mu)\overline{q}$ must vanish, if it vanishes at infinity. The equations

$$\begin{aligned} \text{div}\,(\rho\overline{q}) &= 0 \\ \text{curl}\,(1 - \alpha^2\rho/\mu)\,\overline{q} &= 0 \end{aligned} \qquad (8)$$

are indeed the exact counterparts of the continuity and irrotationality equations in ordinary fluid dynamics. The quantity $\alpha^2\rho/\mu$ is readily identified as $(B^2/\mu\rho)/q^2 = 1/A^2$, or the reciprocal of the Alfven number squared. The same equations could be derived if we had started off with the hydrodynamic analog of Imai[8], and observed that if α is constant, a Kelvin's Theorem would hold; then (8) is deduced in the same way as the continuity and irrotationality equations are deduced in gas dynamics from Euler's equations using the condition of isentropy.

We now consider plane flows[9], and introduce a potential function φ and a stream function ψ in the usual way. Letting q denote the magnitude of the velocity, and θ its inclination to the x-axis, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= (1 - 1/A^2)\, q \cos\theta \\ \frac{\partial \varphi}{\partial y} &= (1 - 1/A^2)\, q \sin\theta \\ \frac{\partial \psi}{\partial x} &= -\rho\, q \sin\theta \\ \frac{\partial \psi}{\partial y} &= \rho\, q \cos\theta \end{aligned} \tag{9}$$

We shall consider flows which are symmetric with respect to the x-axis. In what follows, the subscript $(\ )_o$ denotes quantities on the x-axis (as functions of x alone). The subscript $(\ )_I$ denotes quantities at the first transition ($M^2+A^2 = q^2/(a^2c^2/a^2+c^2) = 1$), the subscript $(\ )_{II}$ denotes quantities at the Alfven (in our case, the second) transition ($A = q/a = 1$), and the subscript $(\ )_{III}$ denotes quantities at the sonic (in our case, the last) transition ($M = q/c = 1$). We shall consider φ and ψ, or rather, convenient functions of each, as independent variables; this permits the flow region being considered, namely in a plane nozzle, to be mapped into a rectangle. (Fig. 1). To this end, we introduce $\xi = \xi(\varphi)$, $\eta = \eta(\psi)$ thus:

$$\begin{aligned} \varphi &= \int_0^{\xi} \left(1 - 1/A_o^2(x) \right) q_o(x)\, dx \\ \psi &= \rho_{III}\, q_{III}\, \eta \end{aligned} \qquad (10)$$

It is clear from the first two equations of (9) that if φ and ψ were used as independent variables directly, the entire flow region in the x-y plane would be mapped into the left half of the φ-ψ -plane covered twice, with a fold at $A = 1$. Introducing ξ avoids this, and makes the mapping one-to-one. η is introduced merely for the convenience of having both independent variables as distances. Physically, ξ is the distance along the x-axis to an equipotential line φ = constant, and η is the distance along the sonic line ($M = 1$) to a streamline ψ = constant. With ξ and η as independent variables, the equations for q, θ, x, and y are:

$$\frac{\partial}{\partial \eta} \left\{ \frac{(1 - 1/A_o^2)\, q_o}{(1 - 1/A^2)\, q} \right\} = - h \frac{\partial \theta}{\partial \xi} \qquad (11)$$

$$\left\{ \frac{(1 - 1/A_o^2)\, q_o}{(1 - 1/A^2)\, q} \right\} \frac{\partial \theta}{\partial \eta} = \frac{\partial h}{\partial \xi} \qquad (12)$$

$$\frac{\partial x}{\partial \eta} = - h \sin \theta \tag{13}$$

$$\frac{\partial y}{\partial \eta} = h \cos \theta \tag{14}$$

where h is defined as $h = \rho_{III}\, q_{III} / \rho q$ i.e., an elementary area ratio.

The equations (11) - (14) are obtained in the usual way, by, e.g., inverting (9) and cross-differentiating. They serve as our basic set of differential equations. It will be seen in Section 4 that the coefficients are analytic functions of ξ, η, q, and θ, thus tending to justify the use of the Cauchy-Kowalewski procedure near the x-axis.

3. POWER SERIES EXPANSIONS

We seek a solution of (11) - (14) in the neighborhood of the axis of symmetry by posing Cauchy data for $q_o(\xi)$, $\theta_o(\xi)$, $x_o(\xi)$, and $y_o(\xi)$ on the axis $\eta = 0$. Evidently, $\theta_o = 0$, $x_o = \xi$, $y_o = 0$.

As the equations (11) - (14) are alternately elliptic and hyperbolic, the Cauchy problem is not well-posed unless $q_o(\xi)$ is analytic. The existence of a solution in some neighborhood of the axis is then guaranteed by the Cauchy-Kowalewski theorem, provided the axis is everywhere non-characteristic. We shall not investigate the region of convergence; otherwise, we can indeed attempt to prove the existence of the solution to (11) - (14) in this manner. In our case, we shall content ourselves to obtain a formal series expansion, the first term of which will turn out to give the "hydraulic approximation". At first sight, we might be concerned with the stability of the flow, since small (non-analytic) variations of the data on the axis can produce large variations in the solution. This, however, does not at all imply the instability of the flow; for in the actual flow, the walls of the nozzle will be fixed, and the small flow perturbations do not correspond to perturbations in the Cauchy data on the axis. This method is valid, therefore, in yielding the contours of the flow nozzle, which would be any of the streamlines in the x-y plane. Thus far, the method described is identical to the Friedrichs procedure[2] for ordinary gas nozzles.

A rather severe difficulty, however, arises at the first transition $M^2+A^2 = 1$. Here, the axis of symmetry is in the characteristic direction. At $A = 1$, similar considerations arise. For the time being, however, we shall ignore those questions and merely obtain the formal series, leaving until Section 4 a more detailed investigation to see how the initial data may, and must, be restricted at these singularities.

We shall naturally give, as known data, the three constants S, α , and E, and the stagnation density $\hat{\rho}$. On account of (7), all the ordinary isentropic gas-dynamical relations (between velocity, Mach number, and the thermodynamic variables) obtain. In particular, the values of all these variables at the three transitions can be determined. We list these in terms of the given data, and shall henceforth treat them as known data also.

Sonic transition:

$$
\begin{aligned}
M_{III} &= 1 \\
q_{III}^2 &= c_{III}^2 = 2\left(\frac{\gamma - 1}{\gamma + 1}\right) E \\
\rho_{III} &= \left[2/(\gamma+1)\right]^{1/(\gamma-1)} \hat{\rho}
\end{aligned}
\tag{15}
$$

Alfven transition:

$$
\begin{aligned}
A_{II} &= 1 \\
\rho_{II} &= \mu/\alpha^2 \\
M_{II}^2 &= (2/\gamma-1) \left[(\alpha^2\hat{\rho}/\mu)^{\gamma-1} - 1\right] \\
c_{II}^2 &= (\gamma-1)\, E\, (\mu/\alpha^2\hat{\rho})^{\gamma-1} \\
q_{II}^2 &= M_{II}^2\, c_{II}^2 = 2E\left[1 - (\mu/\alpha^2\hat{\rho})^{\gamma-1}\right]
\end{aligned}
\tag{16}
$$

First transition:

$$M_I^2 + A_I^2 = 1$$

M_I^2 is given by $\left[1 + (\gamma-1)M_I^2/2\right]^{1/\gamma-1} = (\alpha^2\hat{\rho}/\mu)(1 - M_I^2)$

$$\rho_I = \hat{\rho}\left[1 + (\gamma-1)M_I^2/2\right]^{1/1-\gamma} \tag{17}$$

$$c_I^2 = (\gamma-1)\, E \left[1 + (\gamma-1)M_I^2/2\right]^{-1}$$

$$q_I^2 = M_I^2\, c_I^2$$

Let now $q_o(\xi)$, hence also ρ_o, h_o, M_o^2, etc., be given analytically in ξ. Then we expand in η the variables q, θ, x, and y, depending on whether they are even or odd in η (and with coefficients functions of ξ to be found), thus:

$$\begin{aligned} q(\xi,\eta) &= q_o(1 + \beta_2\eta^2 + \beta_4\eta^4 + \ldots\ldots) \\ \theta(\xi,\eta) &= \theta_1\eta + \theta_3\,\eta^3 + \ldots\ldots \\ x(\xi,\eta) &= \xi + x_2\,\eta^2 + x_4\eta^4 + \ldots\ldots \\ y(\xi,\eta) &= y_1\eta + y_3\,\eta^3 + \ldots\ldots \end{aligned} \tag{18}$$

Now $$c^2/c_o^2 = \frac{1 + \frac{\gamma-1}{2}M_o^2}{1 + \frac{\gamma-1}{2}M^2} = 1 + \frac{\gamma-1}{2}M_o^2 - \frac{\gamma-1}{2}M_o^2(q/q_o)^2$$

$$\rho/\rho_o = (c^2/c_o^2)^{1/\gamma-1} = 1 - M_o^2\beta_2\eta^2 - \frac{M_o^2}{2}(2\beta_4 + \beta_2^2 - (2-\gamma)M_o^2\,\beta_2^2)\,\eta^4 - \ldots$$

$$h/h_o = 1 + (M_o^2-1)\beta_2\eta^2 - \left\{\beta_4(1-M_o^2) - \frac{M_o^2}{2}\beta_2^2\left[3-(2-\gamma)M_o^2\right] - (1 - M_o^2)^2\beta_2^2\right\}\eta^4 - \ldots \tag{19}$$

$$\frac{(1-1/A_o^2)q_o}{(1-1/A^2\ )q} = (\frac{q}{q_o})^{-1}\left(\frac{\rho/\rho_o - A_o^2}{1 \quad - A_o^2}\right)^{-1}$$

$$= 1 - \left(\frac{A_o^2+M_o^2-1}{A_o^2-1}\right)\beta_2\,\eta^2 - \left\{\beta_4\left(\frac{A_o^2+M_o^2-1}{A_o^2-1}\right)\right.$$

$$\left. - \beta_2^2\left[\frac{M_o^2}{1-A_o^2}\left(\frac{3}{2} - \frac{2-\gamma}{2}M_o^2\right) + \left(\frac{A_o^2+M_o^2-1}{A_o^2-1}\right)^2\right]\right\}\eta^4$$

$$- \ldots\ldots$$

Substitution of (18) and (19) into (11) - (14) yields, for the 0-th and 1st powers in ,

$$y_1 = h_o$$

$$\theta_1 = dh_o/d\xi = h_o' \qquad (20)$$

This is evidently the "hydraulic approximation", and it is in no way different from ordinary gas dynamics, as has been noted by Grad[1]. The next higher order terms give

$$\beta_2 = \frac{h_o h_o''}{2}\left(\frac{A_o^2 - 1}{M_o^2+A_o^2-1}\right)$$

$$\theta_3 = \frac{1}{3}\frac{\partial}{\partial\xi}\left(\frac{h_o^2 h_o''}{2}\ \frac{(M_o^2-1)(A_o^2-1)}{M_o^2+A_o^2-1}\right) + \frac{h_o h_o' h_o''}{6}$$

$$y_3 = h_o\left[(M_o^2-1)\beta_2/\ 3 \quad - \theta_1^2/\ 6\right] \qquad (21)$$

$$x_2 = - h_o\ h_o'\ /\ 2$$

(20) and (21) give the first and second order coefficients for the expansions of q, θ, x, and y (equation (18)) near the axis of the nozzle.

Evidently, h_o may not be arbitrarily assigned near the first transition, for β_2 will, in general, become infinite. This is not surprising since the axis is characteristic there, as was pointed out earlier. To see how this initial data must be restricted, we attempted to continue the process to obtain the higher order coefficients, i.e. β_4, β_6 etc., and to observe from these some sufficiency requirement for the β's to remain finite. For example, from the expression of β_2 in (21), h_o'' must evidently be required to vanish at the transition. This procedure, however, led to some rather inconclusive results; the only obvious condition thus deduced was that all higher order derivatives of h_o, from the second onwards, should vanish. Clearly, this is too restrictive. It results in a linear distribution of h_o, which is merely a source or sink. Needless to say, the source or sink flow cannot cross all three transitions, since $M = 1$ is a limiting line for it. In the next section, we study this aspect in greater detail.

We also observe that β_2 (and also β_4, β_6, etc. if we had continued on) will vanish at $A_o = 1$. This merely says that the equipotential line $A_o = 1$ is also a constant speed line; thus the line $A_o = 1$ and the line $A = 1$ coincide, as can be seen already from equation (9).

In section 5, we shall give some numerical examples, and also give some more physical interpretations of these results.

4. MORE DETAILED INVESTIGATION OF THE EQUATIONS AND THE SINGULARITIES

The basic equations (11) - (14), while convenient for calculating the power series expansions, do not reveal the nature of the singularities. We re-write (11) and (12) (ignoring the harmless and uncoupled equations (13) and (14)), using $dp = c^2 d\rho$, into

$$\left(1-\frac{1}{A_o^2}\right) q_o \left(\frac{M^2+A^2-1}{A^2}\right) \frac{\partial q}{\partial \eta} = \left(1-\frac{1}{A^2}\right)^2 q^2 h \frac{\partial \theta}{\partial \xi} \qquad (22)$$

$$\left(1-\frac{1}{A_o^2}\right) q_o \frac{\partial \theta}{\partial \eta} = \left(1-\frac{1}{A^2}\right) h (M^2-1) \frac{\partial q}{\partial \xi} \qquad (23)$$

The characteristic directions are given by

$$\left(\frac{d\xi}{d\eta}\right)^2 = \frac{(A^2-1)^3 \ (M^2-1)}{(A_o^2-1)^2 \ (M^2+A^2-1)} \ \frac{A_o^4 q^2 h^2}{A^4 q_o^2} \qquad (24)$$

so that the axis becomes characteristic at $M^2+A^2 = 1$, and (seemingly) also at $A_o = 1$. The sonic transition, $M = 1$, offers no difficulty, since there the axis is perpendicular to the characteristic directions, and hence is noncharacteristic. We handle each singularity separately. Fig. 2(a) shows the characteristics in the nozzle flow.

(1) $M^2+A^2 = 1$

The characteristics here are tangent to the axis, and the transition line is perpendicular to it. To illuminate the situation, we consider the Tricomi equation

$$xu_{yy} + u_{xx} = 0 \tag{25}$$

which has characteristics as shown in fig. 2(b). The x-y axes are the counterparts of the ξ-η axes in our problem, the origin corresponds to our singularity, and the y-axis is the transition line (although in our problem the transition line is not the η-axis, but is only tangent to it at the singularity). Cauchy data on the x-axis, i.e. $u(x,0)$ and $u_y(x,0)$, cannot be arbitrarily posed at $x = 0$.

We restrict our attention only to solutions symmetric about the x-axis. Since the y-axis is non-characteristic, we may consider every analytic solution in the neighborhood of the origin as the (unique) solution of a Cauchy problem with analytic data $u(0,y)$ and $u_x(0,y)$ on the y-axis. Thus, if we give arbitrary analytic data (symmetrically) on the y-axis, we shall be able to determine the form of every symmetric analytic solution in the neighborhood of the origin, and in particular, its behavior on the x-axis at the origin. This will, in turn, enable us to pose the right kind of Cauchy data on the x-axis.

Thus, giving

$$u(0,y) = a_o + a_1 y^2 + a_2 y^4 \quad . \; . \; . \qquad (26)$$
$$u_x(0,y) = b_o + b_1 y^2 + b_2 y^4 \quad . \; . \; .$$

substituting into (25) a power series solution satisfying (26) and computing coefficients term by term, we find on the x-axis,

$$u(x,0) = a_o + b_o x - (a_1/3)x^3 - (b_1/6)x^4 + (2a_2/15)x^6 ---- \qquad (27)$$

and, naturally, $u_y(x,0) = 0$

The form of (27) suggests that the data u(x,0) to be posed on the x-axis should be a power series with the 2nd, 5th, 8th, 11th,--- powers missing, thus:

$$u(x,0) = A_o + A_1 x + A_3 x^3 + A_4 x^4 + A_6 x^6 --- \qquad (28)$$

Indeed, if we take any convergent power series of this form as data on the x-axis, and repeat the formal procedure, we shall find a formal solution u(x,y). On the y-axis, this formal solution u(o,y) and its derivative $u_x(0,y)$ indeed do converge (with a smaller radius of convergence than u(x,0)). Therefore, applying the Cauchy-Kowalewski Theorem, the formal solution u(x,y) is convergent, and is an actual solution, in the neighborhood of the origin.

This proves that Cauchy data in the form of (28) are admissible across the origin.

The same procedure will now be applied to our problem. Let the origin of the $\xi - \eta$ plane correspond to the intersection of the first transition with the axis. We give arbitrary analytic data on $\xi = 0$, and determine the behavior of the (analytic) solution on $\eta = 0$; this in turn gives us the form of analytic data to be posed on $\eta = 0$, as is needed in the problem. Specifically, we want the behavior of $h(\xi, 0)$, which is needed in (20) and (21).

As before, for any function $f(\xi, \eta)$, f_o denotes $f(\xi, 0)$. Correspondingly f^o shall denote $f(0, \eta)$. Evidently $f^o_o = f(0,0) = f_I$.

We give

$$q^o = q_I \, (1 + a_2 \eta^2 + a_4 \eta^4 + \ldots) \qquad\qquad (29)$$
$$\theta^o = b_1 \eta + b_3 \eta^3 + b_5 \eta^5 + \ldots$$

and seek for solutions

$$q = q^o \, (1 + A_1 \xi + A_2 \xi^2 + A_3 \xi^3 + \ldots) \qquad\qquad (30)$$
$$\theta = \theta^o \, (1 + B_1 \xi + B_2 \xi^2 + B_3 \xi^3 + \ldots)$$

where A_1, A_2, ..., B_1, B_2, ... are functions of η alone, and are to be determined.

In what follows, we shall let the adiabatic constant γ be 3/2. This simplifies the algebraic manipulations considerably. The conclusions reached, however, are also valid for other values of γ. Expanding all the other variables, we have

$$\frac{\rho}{\rho^o} = \left\{ 1 + \frac{\gamma-1}{2} M^{o2} \left[1 - \left(\frac{q}{q^o}\right)^2 \right] \right\}^{\frac{1}{\gamma-1}}$$

$$= 1 - M^{o2} A_1 \xi + C_2 \xi^2 + C_3 \xi^3 + \ldots\ldots$$

$$\frac{h}{h^o} = \frac{q^o}{q} \frac{\rho^o}{\rho} = 1 - D_1 \xi - D_2 \xi^2 - D_3 \xi^3 - \ldots\ldots \qquad (31)$$

$$(1 - \frac{1}{A^2})\, q = (1 - \frac{\alpha^2 \rho}{\mu})\, q = \left\{ 1 - \frac{1}{A^{o2}} \left(\frac{\rho}{\rho^o} \right) \right\} q$$

$$= (1 - \frac{1}{A^{o2}}) q^o \left[1 + E_1 \xi + E_2 \xi^2 + E_3 \xi^3 + \ldots\ldots \right]$$

$$\frac{(1 - \frac{1}{A_o^2}) q_o}{(1 - \frac{1}{A^2})\, q} = \frac{(1 - \frac{1}{A_I^2}) q_I}{(1 - \frac{1}{A^{o2}}) q^o} (1 + E_{20}\xi^2 + E_{30}\xi^3 + \ldots)(1 + E_1\xi + E_2\xi^2 + \ldots)^{-1}$$

$$= G \left[1 - E_1\xi + (E_{20} - E_2 + E_1^2)\xi^2 + (E_{30} - E_3 + 2E_1E_2 - E_1E_{20} - E_1^3)\xi^3 + \ldots \right]$$

where:

$$C_2 = \frac{M^{o4}}{4} A_1^2 - \frac{M^{o2}}{2} (A_1^2 + 2A_2)$$

$$C_3 = \frac{M^{o4}}{4} A_1 (A_1^2 + 2A_2) - M^{o2} (A_1 A_2 + A_3)$$

$$C_4 = \frac{M^{o4}}{16} (A_1^2 + 2A_2)^2 + \frac{M^{o4}}{2} A_1 (A_1 A_2 + A_3) - \frac{M^{o2}}{2} (A_2^2 + 2A_1A_3 + 2A_4)$$

$$D_1 = (1 - M^{o2})\, A_1$$

$$D_2 = (1 - M^{o2})\, A_2 - A_1^2 (1 - \frac{M^{o2}}{2} + \frac{3}{4} M^{o4}) \qquad (32)$$

$$D_3 = A_3(1 - M^{o2}) - 2\,A_1A_2\left(1 - \frac{M^{o2}}{2} + \frac{3}{4}M^{o4}\right) + A_1{}^3\left(1 - \frac{M^{o2}}{2} - \frac{M^{o6}}{2}\right)$$

$$E_1 = A_1\left(\frac{M^{o2} + A^{o2} - 1}{A^{o2} - 1}\right) \quad ; \quad E_{10} = \Big(E_1\Big)_{\eta = o} = 0$$

$$E_2 = A_2\left(\frac{M^{o2} + A^{o2} - 1}{A^{o2} - 1}\right) + \frac{A_1{}^2\,M^{o2}}{A^{o2} - 1}\left(\frac{3}{2} - \frac{M^{o2}}{4}\right) \; ; \; E_{20} = \frac{A_{10}M_I{}^2}{A_I^2 - 1}\left(\frac{3}{2} - \frac{M_I{}^2}{4}\right)$$

$$E_3 = A_3\left(\frac{M^{o2} + A^{o2} - 1}{A_o{}^2 - 1}\right) + \frac{A_1A_2\,M^{o2}}{A^{o2} - 1}\left(3 - \frac{M^{o2}}{2}\right) + \frac{M^{o2}\,A_1{}^3}{2(A^{o2} - 1)}(1 - M^{o2}) \;;$$

$$E_{30} = \frac{A_{10}\,A_{20}\,M_I{}^2}{A_I^2 - 1}\left(3 - \frac{M_I{}^2}{2}\right) + \frac{M_I{}^2\,A_{10}{}^3}{2(A_I{}^2 - 1)}(1 - M_I^2)$$

$$G = \frac{\left(1 - \dfrac{1}{A_I^2}\right)q_I}{\left(1 - \dfrac{1}{A^{o2}}\right)q^o}$$

Substituting (30) and (31) into (11) and (12), and equating the coefficients for each power of ξ, we obtain:

$$\left.\begin{aligned} A_1 &= -\frac{1}{h^o(1 - M^{o2})}\,G\,\frac{d\theta^o}{d\eta} \\ B_1 &= -\frac{1}{h^o\,\theta^o}\,\frac{dG}{d\eta} \\ B_2 &= \frac{1}{2h^o\theta^o_o}\cdot\frac{d}{d\eta}(E_1G) + \frac{1}{2}(1 - M^{o2})A_1B_1 \end{aligned}\right\} \qquad (33)$$

$$\left. D_2 = -\frac{1}{2h^o}\left[G\,\frac{d}{d\eta}(B_1\theta^o) - GE_1\,\frac{d\theta^o}{d\,\eta}\right] \right\} \qquad (34)$$

$$D_3 = -\frac{1}{3h^o}\left[G\frac{d}{d\eta}(B_2\theta^o) - GE_1\frac{d}{d\eta}(B_1\theta^o) + G(E_{20} - E_2 + E_1{}^2)\frac{d\theta^o}{d\eta}\right]$$

We are after the behavior of $h(\xi, 0)$; hence we must find D_{20} and D_{30}. With q^o and θ^o given by (29), we have, from (20),

$$G = 1 - a_2\left(\frac{A_I{}^2 + M_I{}^2 - 1}{A_I{}^2 - 1}\right)\eta^2 - \left\{a_4\left(\frac{A_I{}^2 + M_I{}^2 - 1}{A_I{}^2 - 1}\right) - a_2{}^2\left(\frac{M_I{}^2}{1 - A_I{}^2}\right)\left(\frac{3}{2} - \frac{M_I{}^2}{4}\right) + \ldots\right\}\eta^4$$

$$= 1 + \frac{1}{2}\left(3 - \frac{M_I{}^2}{2}\right)\left(\frac{M_I{}^2}{1 - A_I{}^2}\right)a_2{}^2\eta^4 + \ldots\ldots \tag{35}$$

Thus

$$-2h_I D_{20} = \left[G\frac{d}{d\eta}(B_1\theta^o) - GE_1\frac{d\theta^o}{d\eta}\right]_{\eta=0}$$

$$= 1\cdot\left(\frac{d}{d\eta}\left(-\frac{1}{h^o}\frac{dG}{d\eta}\right)\right)_{\eta=o} - GE_{10}\left(\frac{d\theta^o}{d\eta}\right)_{\eta=o} = 0 \tag{36}$$

as anticipated. Now

$$-3h_I D_{30} = \left(\frac{d}{d\eta}(B_2\theta^o)\right)_{\eta=o}$$

all other terms vanishing since $E_{10} = 0$.

Using B_2 from (33),

$$-6h_I D_{30} = \left(\frac{d}{d\eta}\left(\frac{1}{h^o}\right)\right)_{\eta=o}\left(\frac{d}{d\eta}(E_1 G)\right)_{\eta=o} + \frac{1}{h_I}\left(\frac{d^2}{d\eta^2}(E_1 G)\right)_{\eta=o}$$

$$+ (1 - M_I{}^2)A_{10}\left(\frac{d}{d\eta}B_1\theta^o\right)_{\eta=o} + \left(B_1\theta^o\right)_{\eta=o}\left(\frac{d}{d\eta}(1 - M^{o2})A_1\right)_{\eta=o}$$

But

$$\left(\frac{d}{d\eta}(E_1 G)\right)_{\eta=o} = E_{10}\left(\frac{dG}{d\eta}\right)_{\eta=o} + 1\cdot\left(\frac{dE_1}{d\eta}\right)_{\eta=o}$$

$$= \left(\frac{d}{d\eta}A_1\left(\frac{M^{o2} + A^{o2} - 1}{A^{o2} - 1}\right)\right)_{\eta=o} = \left(\frac{d}{d\eta}\left(\frac{G}{h^o}\right)\left(\frac{d\theta^o}{d\eta}\right)\frac{M^{o2} + A^{o2} - 1}{(M^{o2} - 1)(A^{o2} - 1)}\right)_{\eta=o}$$

$$= 0$$

since the quantity being differentiated is an even function in η . Similarly,

$$\left(\frac{d^2}{d\eta^2}(E_1G)\right)_{\eta=0} = E_{10}\left(\frac{d^2G}{d\eta^2}\right)_{\eta=0} + 2\left(\frac{dE_1}{d\eta}\right)\left(\frac{dG}{d\eta}\right)_{\eta=0} + 1\cdot\left(\frac{d^2E_1}{d\eta^2}\right)_{\eta=0}$$

$$= \left(\frac{d^2}{d\eta^2}\left[\frac{G}{h^o(M^{o2}-1)}\frac{d\theta^o}{d\eta}\left(\frac{M^{o2}+A^{o2}-1}{A^{o2}-1}\right)\right]\right)_{\eta=0}$$

$$= \left(\frac{G}{h^o}\frac{d\theta^o}{d\eta}\right)_{\eta=0}\left(\frac{d^2}{d\eta^2}\left(\frac{M^{o2}+A^{o2}-1}{(M^{o2}-1)(A^{o2}-1)}\right)\right)_{\eta=0} + \cdots\cdots$$

$$= \frac{b_1}{h_I}\left(\frac{d^2}{d\eta^2}\left(\frac{M^{o2}+A^{o2}-1}{(M^{o2}-1)(A^{o2}-1)}\right)\right)_{\eta=0}$$

With q^o given by (29), and ρ^o, c^{o2} in terms of q^o given by (19) and (20) respectively, we readily find that

$$M^{o2} + A^{o2} = M_I^2 + A_I^2 + \left(M_I^2A_I^2 + \left(2 + \frac{M_I^2}{2}\right)M_I^2\right)a_2\eta^2 + \cdots\cdot$$

and that $\left(\frac{d^2}{d\eta^2}\left(\frac{M^{o2}+A^{o2}-1}{(M^{o2}-1)(A^{o2}-1)}\right)\right)_{\eta=0} = \frac{1}{(M_I^2-1)(A_I^2-1)}\left(\frac{d^2}{d\eta^2}(M^{o2}+A^{o2})\right)_{\eta=0}$

$$= \frac{1}{(M_I^2-1)(A_I^2-1)}\left(A_I^2M_I^2+\left(2+\frac{M_I^2}{2}\right)M_I^2\right)2a_2 = 0$$

Thus, D_{30} is non-zero, and is moreover independent of D_{10} (which is equal to $\left[(M^{o2}-1)A_1\right]_{\eta=0} = \left[\frac{G}{h^o}\frac{d\theta^o}{d\eta}\right]_{\eta=0} = \frac{b_1}{h_I}$), since b_1 and a_2 are both arbitrary and thus independent. Thus, we see that the data to be given on the ξ axis must be of the form

$$h(\xi,0) = h_I + D_{10}\xi + D_{30}\xi^3 + \cdots\cdot \qquad (37)$$

For h_o of this form, equations (20) and (21) will then give us the first and second approximations of the flow streamlines, without any further difficulty from the singularity at $M^2 + A^2 = 1$.

We shall content ourselves with choosing third order polynomials, for h_o, since investigation of higher order terms becomes algebraically prohibitive. A third order polynomial h_o will give a flow crossing all three transitions, since such a function can first decrease and then increase, thus giving convergent-divergent nozzles. All higher order coefficients, moreover, can also be determined, in principle, and will be finite everywhere. Unlike the case of the Tricomi equation, unfortunately, we have no proof that if h_o is a convergent series, the resulting formal solution will also be convergent, and thus be a true solution in the neighborhood of the origin. This gives yet another reason why our solution remains a formal solution.

If we had taken higher order terms in the formal expansion given in section 3, we would obtain, as the next higher term corresponding to (21)

$$\beta_4 = \frac{1}{4} \frac{A_o^2 - 1}{M_o^2 + A_o^2 - 1} \left\{ - \frac{M_o^2}{8} h_o^2 h_o''^2 \left(3 - \frac{M_o^2}{2}\right) \frac{A_o^2 - 1}{(M_o^2 + A_o^2 - 1)^2} + \frac{h_o^2 h_o''}{4} + \frac{h_o h_o''^2}{2} \frac{(M_o^2 - 1)(A_o^2 - 1)}{M_o^2 + A_o^2 - 1} + h_o \theta_3' \right\} \qquad (38)$$

where θ_3 is given by (21). Near the singularity, $\frac{h_o''}{A_o^2 + M_o^2 - 1}$ tends to a finite limit. Hence the second and third terms in the bracket tend to zero. If h_o has been chosen according to (37), the first and last terms in the bracket will, undoubtedly, have their zero order terms (in ξ) cancel, so that the entire expression in the bracket starts off with first power in ξ . Thus β_4 will be finite. However, direct verification of this is difficult because of extremely cumbersome algebraic manipulations.

R. Seebass[10] has obtained a special exact solution across the first transition (hypercritical transition, in his terminology) for $\gamma = 2$. The resulting $h_o(x)$ from his solution is exactly of the form (37). Kogan[11] has also given a special solution across this transition. His $h_o(x)$ has an x^2-term, which seemingly negates our result. Kogan's solution, however, is approximate, and is valid only to linear terms in x. Hence, no information may be extracted from it for the behavior of higher order terms.[12]

(2) $A = 1$

Equation (24) seems to indicate that the axis is again characteristic at $A_o = 1$. This, however, is not the case. Consider the line on which $A_o(\xi) = 1$. This is an equipotential line. But (9) shows this also coincides with the line $A = 1$; this fact has been observed earlier at the end of section 3. Therefore, in (24), the characteristic direction is $d\xi/d\eta = 0$ at this singularity, which implies that the axis is perpendicular to the characteristic direction, and is thus non-characteristic. The data given on the axis, therefore, should not require any special handling as needed at the other singularity $M^2 + A^2 = 1$.

However, as a result of $A_o = 1$ (and $A = 1$ simultaneously), (22) and (23) indeed fail to satisfy the hypotheses of the standard Cauchy-Kowalewski Theorem at this singularity. For, in applying the Cauchy-Kowalewski Theorem, A_o must be considered as a function of the independent variable ξ, while A is considered as a function of the dependent variable q, and the derivatives with respect to η must be solvable from the equation without recourse to the fact that $A(\xi, 0) = A_o$; i.e., the equation must have analytic coefficients and be non-characteristic along the initial line independently of the Cauchy data. Nevertheless, the coefficients of the formal expansion are finite and apparently reasonable; this leads us to believe that the singularity is actually harmless in nature, and does not affect the validity of the solution. A refined mathematical investigation of the well-posed nature of the Cauchy problem near this singularity will not be attempted in this paper.

5. NUMERICAL EXAMPLES.

We shall now give two examples of flows calculated by the method presented in the previous sections. Since we are interested in mathematical properties rather than practical usefulness in this paper, we take 3rd order polynomials for h_o as required in Section 4 to insure well behaved solutions; the nozzles obtained, however, are rather long and not too practical for actual use.

Nozzle "A" is calculated with given

$$h_o = 5 - 6\xi + 2\xi^3$$

and nozzle "B" with

$$h_o = 3 - 3\xi + \xi^3$$

Once we have h_o, and the fact that $M_o{}^2 + A_o{}^2 = 1$ at $\xi = 0$, all the other flow quantities on the axis are completely known. In particular, we easily obtain for $\gamma = 1.5$,

$$h^2 = \frac{1}{M^2} \left(\frac{M^2+4}{5} \right)^5$$

which permits the determination of $M_o(\xi)$. For $\xi = 0$, we then have M_I^2 and $A_I^2 = 1-M_I^2$. But since A^2 is inversely proportional to ρ, we immediately have

$$A^2 = A_I{}^2 \left(\frac{1 + \frac{\gamma-1}{2} M^2}{1 + \frac{\gamma-1}{2} M_I{}^2} \right)^{\frac{1}{\gamma-1}}$$

We shall confine our interest to the region $-0.5 \leq \xi \leq 1.5$, which is the length of the nozzle. At $\xi = 1.5$, the flow is supersonic (hyperbolic), and the nozzle contour beyond that can be easily shaped by simple wave theory. On the other hand, at the inlet, we do not have a flow at rest, but only a very low speed flow with a given speed distribution across the inlet section. That this can be physically realized we shall assume.

We also observe that when the flow is at rest in the reservoir, $M = 0$, but $A \neq 0$, since $A^2 = \frac{\mu}{a^2 \hat{\rho}}$. We denote this by $\hat{A}^2$. For each nozzle, we have the following constants:

Nozzle	M_o inlet	M_o exit	M_I	A^2_I	$\hat{A}^2$
"A"	0.074	2.66	0.123	.987	0.980
"B"	0.131	2.19	0.195	0.962	0.943

Using equations (20) and (21), the flows are computed and shown in figs. 3 and 4 respectively. The first approximation (hydraulic theory) is superimposed on the same figure. The close agreement between the two approximations tends to suggest probable convergence of the formal power series. In fig. 3 (Nozzle A) streamlines are shown up to $\eta = 0.1$, and in fig. 4 (Nozzle B), up to $\eta = 0.2$ This means that the throat-length ratios are, respectively, 10% and 20%, a rather long nozzle in each case. Larger values in η result in wide difference between the second and first approximations, suggesting that higher order terms must be included, or even, perhaps, that the formal solution is no longer convergent.

The positions of the transition lines are of interest. They are also shown in figs. 3 and 4. Their locations can be qualitatively seen by the simple considerations as follows: The line $A = 1$ is, as mentioned before, an equipotential line as well as a constant-speed line, so its location is obviously coincident with $\xi = \xi_{II}$. In the vicinity of the line $M^2 + A^2 = 1$, or $\xi = \xi_I$, the quantity β_2 is negative. This implies that in this vicinity, the speed decreases away from the axis on an equipotential line. A constant-speed line, therefore, must bend forward (i.e. downstream) of an equipotential line. Hence the transition line $M^2+A^2 = 1$ bends dowstream from the equipotential line $\xi_I = 0$. An identical argument shows that the sonic transition $M = 1$ bends backwards (or upstream) away from the axis; this is the same as in ordinary gas nozzles.

We conclude with a remark comparing our results with the flow of an ordinary non-conducting gas. The latter can be computed either by the procedure of Friedrichs for plane flow, or simply by setting the magnetic field to zero in our results (setting $\alpha = 0$). We have already observed that the first approximation (hydraulic theory) remains the same. For the second approximation (21), β_2 simply becomes $h_o h_o''/2$. y_3 changes accordingly, while x_2 remains the same. In our examples, the flows will not be widely different, since the second approximation is close to the first.

A more detailed study of the effect of the field on the flow, involving more numerical calculations, as well as the more practical aspects of these flows (e.g. comparison with other field orientations, practical realizability, etc.), will be considered in a subsequent work.

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FOOTNOTES

1. H. Grad, Rev. of Mod. Phys., 32, 830 (1960).

2. K. O. Friedrichs, "Theoretical Studies on Flow Through Nozzles and Related Problems", Nat. Def. Res. Comm., Appl. Math. Panel, Report 82.1 R (1944).

3. F. I. Frankl, Izvestiya Akad. Nauk SSSR, Ser. Matem., 9, 387 (1945).

4. The situation of more equations than unknowns is frequently encountered in physical problems, e.g., curl $\overline{q} = 0$, div $\overline{q} = 0$ in classical hydrodynamics.

5. M. N. Kogan, Prik. Mat. Mekh., 23, 70 (1959).

6. E. L. Resler, Jr., and W. R. Sears, Jour. of Fluid Mech., 5, 257 (1959).

7. If the field and velocity are always in one plane (two-dimensional flow in the restricted sense), then parallel velocity and field at one section (say, reservoir or infinity) implies parallel velocity and field everywhere.

8. I. Imai, Rev. of Mod. Phys., 32, 992 (1960).

9. We could just as well have considered axi-symmetric flow, with a slight increase in complication, and obtained a probably more useful result. Nevertheless, the essential mathematical features are no different, and we shall do the simpler case.

10. R. Seebass, Quat. of Appl. Math. (1961), to appear.

11. M. N. Kogan, Prik. Mat. Mekh., 25, 132 (1961).

12. The author heartily thanks Mr. R. Seebass of Cornell University for an interesting discussion of these results.

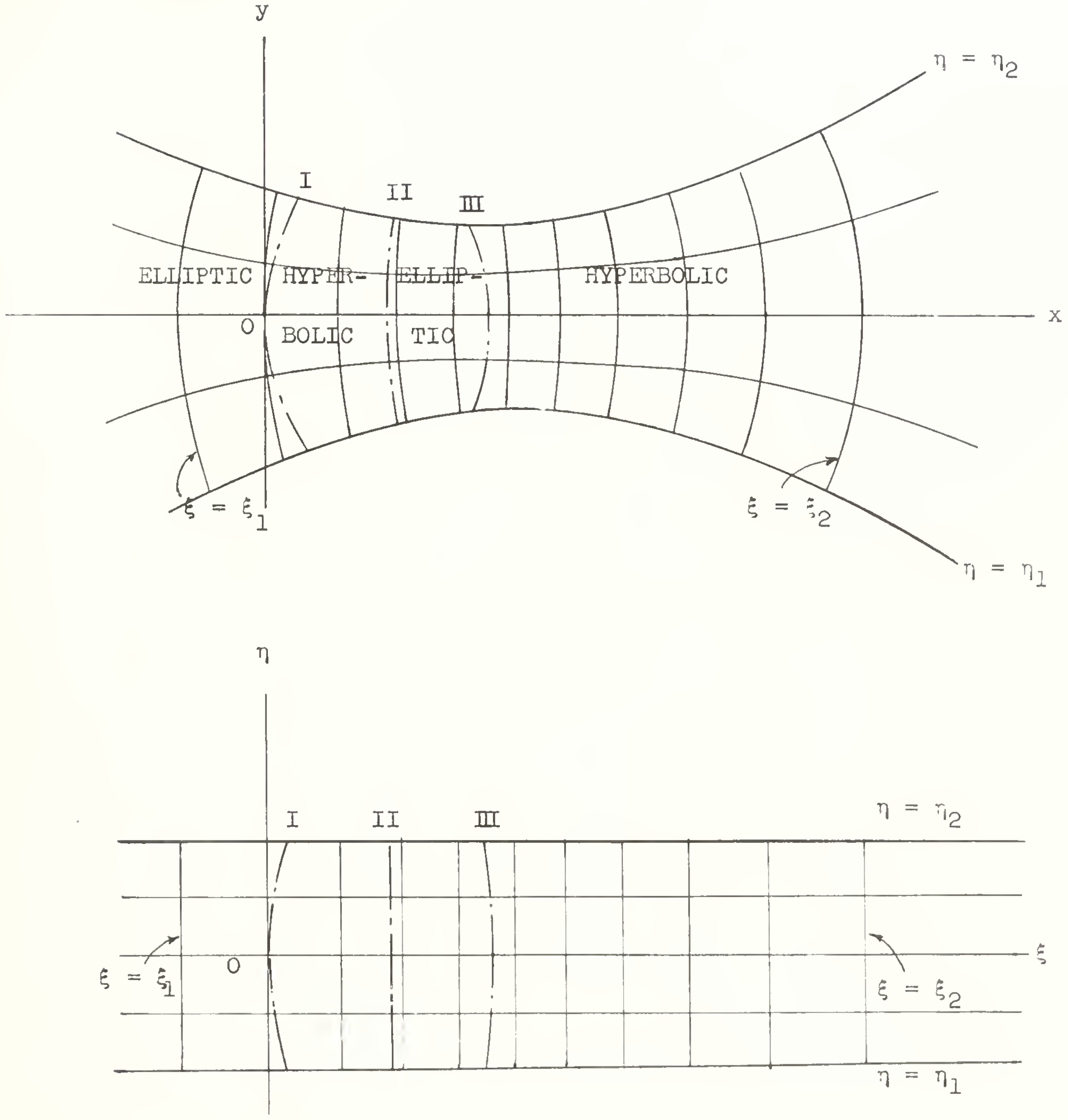

FIG. 1. Nozzle Flow in the x-y plane mapped into a rectangle in the ξ-η plane.

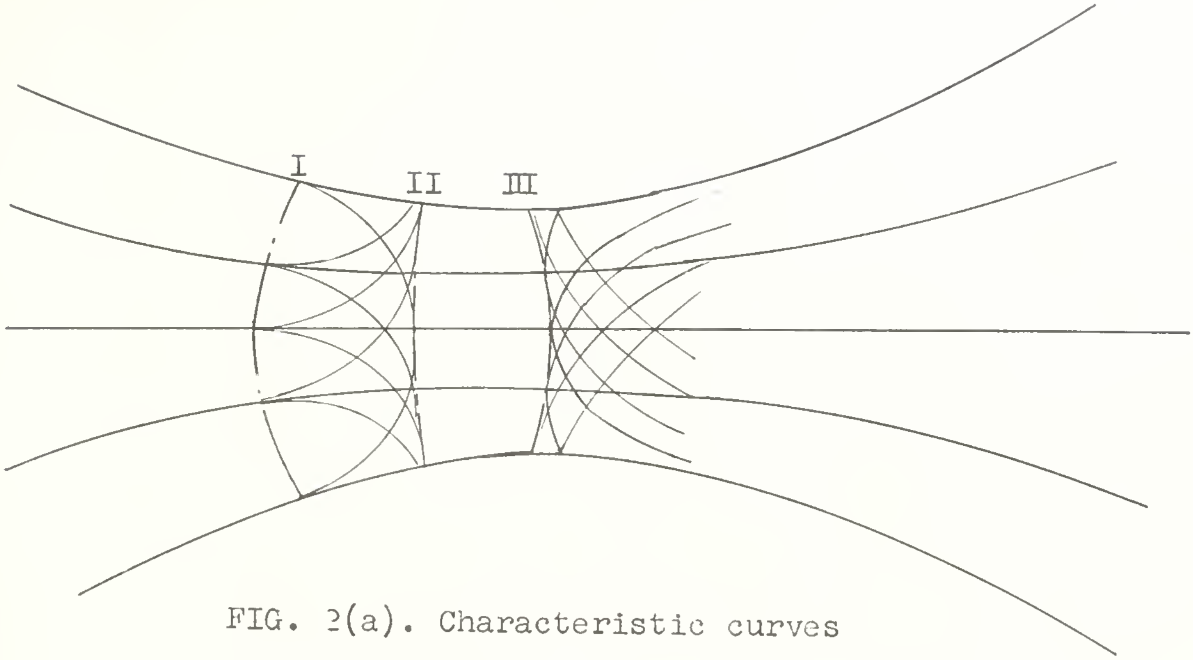

FIG. 2(a). Characteristic curves in nozzle flow.

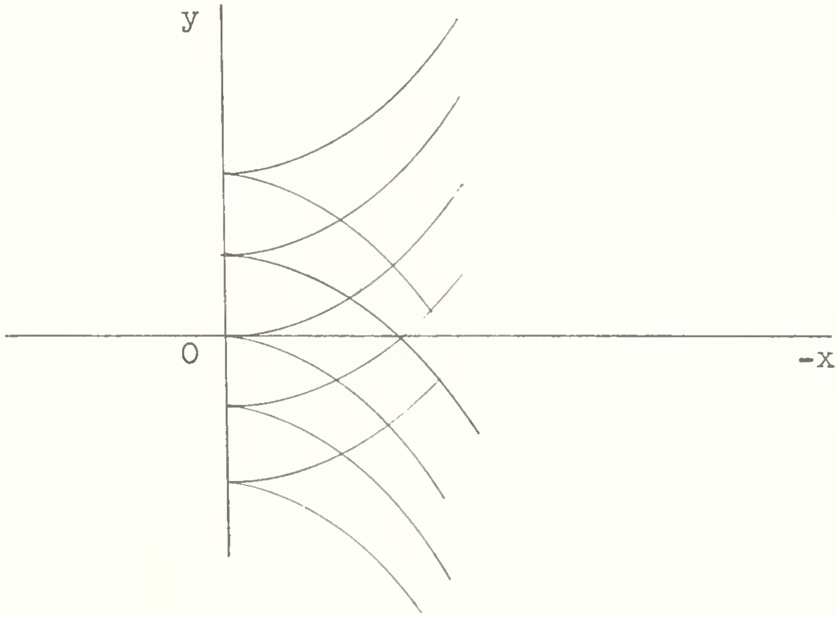

FIG. 2(b). Characteristic curves for Tricomi equation $xu_{yy} + u_{xx} = 0$.

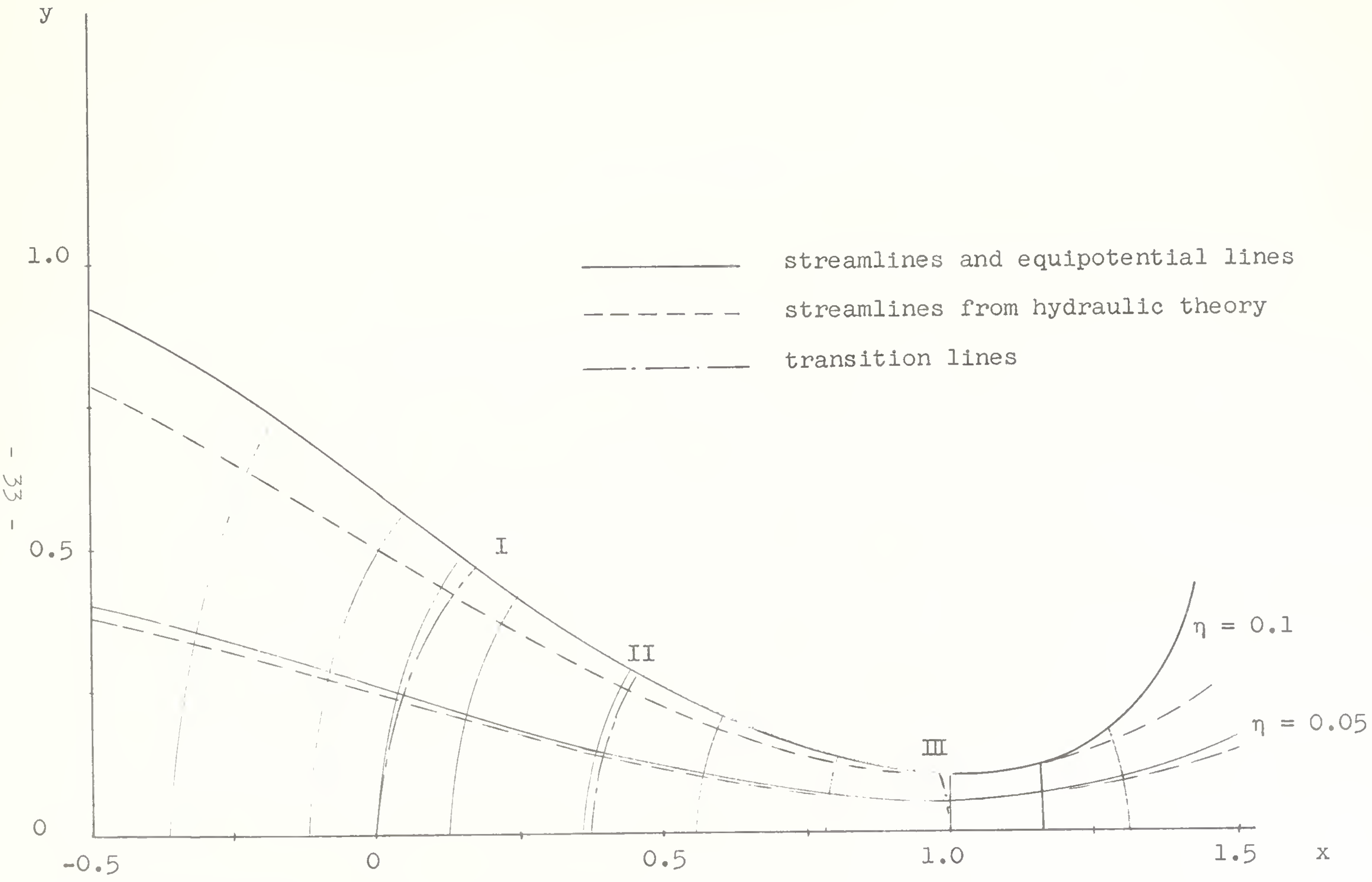

FIG . 3. Streamlines, equipotential lines, and transition lines for Nozzle "A", with $h_o = 5 - 6\xi + 2\xi^3$.

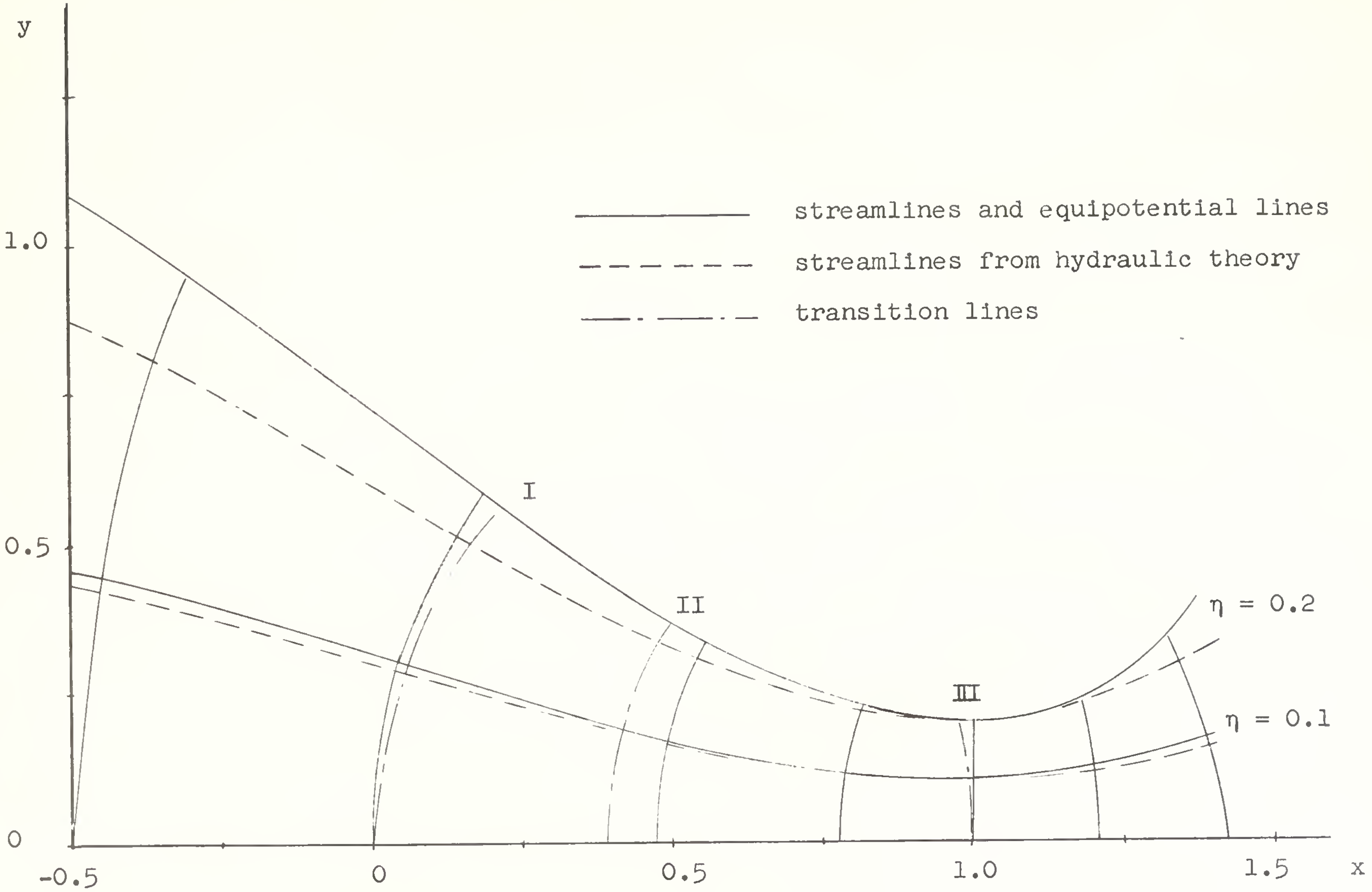

FIG. 4. Streamlines, equi-potential lines, and transition lines for Nozzle "B", with $h_o = 3 - 3\xi + \xi^3$.

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